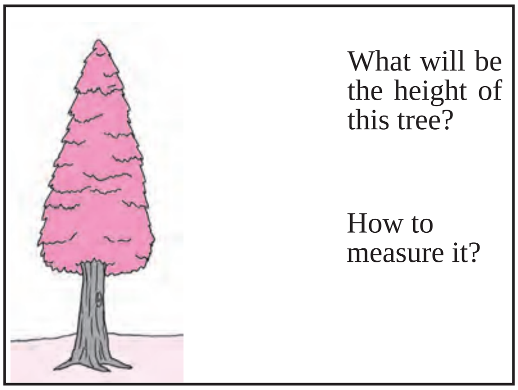
8 Trigonometry

Let's study.

- Introduction of Trigonometry
- Trigonometic Ratios
- Relations among Trigonometric Ratios
- Trigonometric Ratios of Particular Angles

Introduction to Trigonometry



We can measure distances by using a rope or by walking on ground, but how to measure the distance between a ship and a light house? How to measure the height of a tall tree?

Observe the above pictures. Questions in the pictures are related to mathematics. Trigonometry, a branch of mathematics, is useful to find answers to such questions. Trigonometry is used in different branches of Engineering, Astronomy, Navigation etc.

The word Trigonometry is derived from three Greek words 'Tri' means three, 'gona' means sides and 'metron' means measurements.

Let's recall.

We have studied triangle. The subject trigonometry starts with right angled triangle, theorem of Pythagoras and similar triangles, so we will recall these topics.

- In Δ ABC, $\angle$B is a right angle and side AC opposite to $\angle$B, is hypotenuse. Side opposite to $\angle$A is BC and side opposite to $\angle$C is AB.
  Using Pythagoras' theorem, we can write the following statement for this triangle.
  $(AB)^2 + (BC)^2 = (AC)^2$

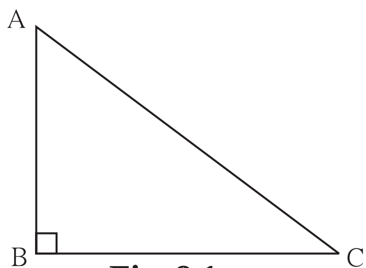


Fig. 8.1

- If $\Delta ABC \sim \Delta PQR$ then their corresponding sides are in the same proportions.

  So $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$

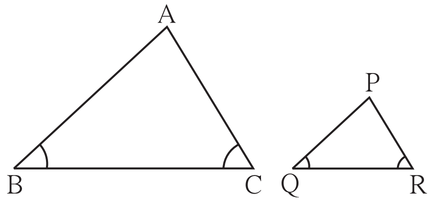


Fig. 8.2

Let us see how to find the height of a tall tree using properties of similar triangles.

Activity : This experiment can be conducted on a clear sunny day.

Look at the figure given alongside.

Height of the tree is QR, height of the stick is BC.

Thrust a stick in the ground as shown in the figure. Measure its height and length of its shadow. Also measure the length of the shadow of the tree. Rays of sunlight are parallel. So Δ PQR and Δ ABC are equiangular, means similar triangles. Sides of similar triangles are proportional.

So we get $\frac{QR}{PR} = \frac{BC}{AC}$.

Therefore, we get an equation,

height of the tree = QR = $\frac{BC}{AC} \times PR$

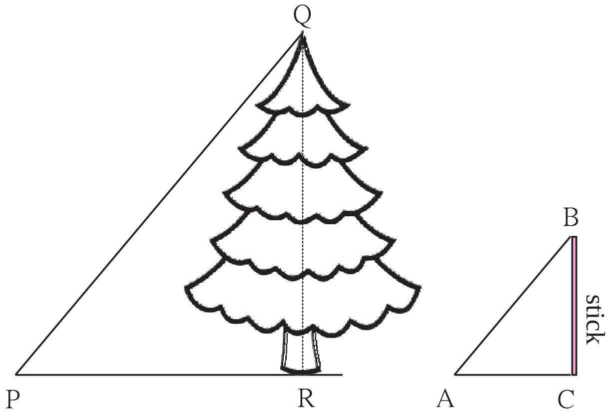


Fig.8.3

We know the values of PR, BC and AC. Substituting these values in this equation, we get length of QR, means height of the tree.

Use your brain power !

It is convenient to do this experiment between 11:30 am and 1:30 pm instead of doing it in the morning at 8'O clock. Can you tell why ?

Activity : You can conduct this activity and find the height of a tall tree in your surrounding. If there is no tree in the premises then find the height of a pole.

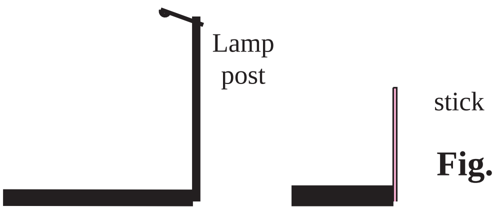


Fig. 8.4

Let's learn.

Terms related to right angled triangle

In right angled Δ ABC, $\angle$B = 90°, $\angle$ A and $\angle$ C are acute angles.

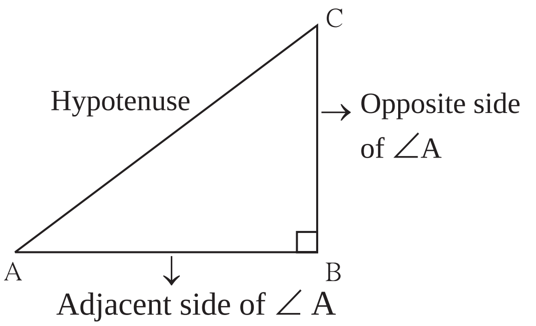


Fig. 8.5

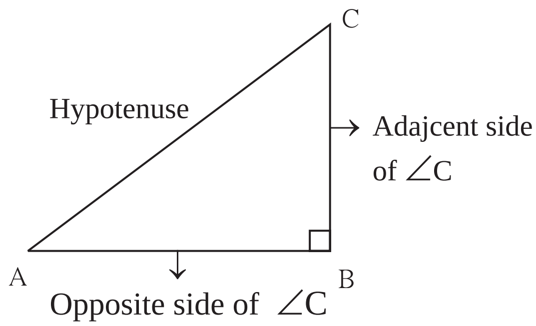


Fig. 8.6

Ex. In the figure 8.7, Δ PQR is a right angled triangle. Write-

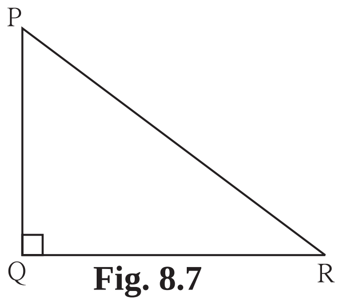


Fig. 8.7

side opposite to $\angle$ P =.....

side opposite to $\angle$ R =.....

side adjacent to $\angle$ P = ...

side adjacent to $\angle$ R = ...

Trigonometic ratios

In the adjacent Fig.8.8 some right angled triangles are shown. $\angle$B is their common angle. So all right angled triangles are similar.

Δ PQB ~ Δ ACB

$$\therefore \frac{PB}{AB} = \frac{PQ}{AC} = \frac{BQ}{BC}$$

$$\therefore \frac{PQ}{AC} = \frac{PB}{AB} \quad \therefore \frac{PQ}{PB} = \frac{AC}{AB} \text{ ... alternando}$$

$$\frac{QB}{BC} = \frac{PB}{AB} \quad \therefore \frac{QB}{PB} = \frac{BC}{AB} \text{ ... alternando}$$

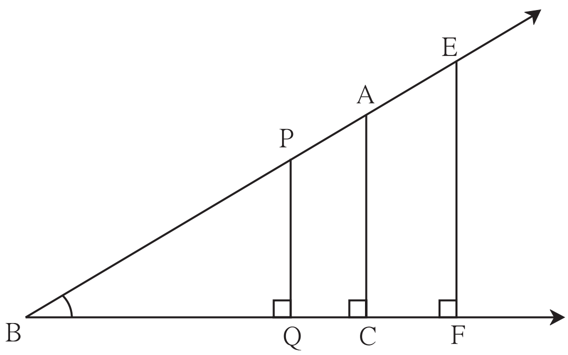


Fig. 8.8

The figures of triangles in 8.9 and.8.10 are of the triangles separated from the figure 8.8.

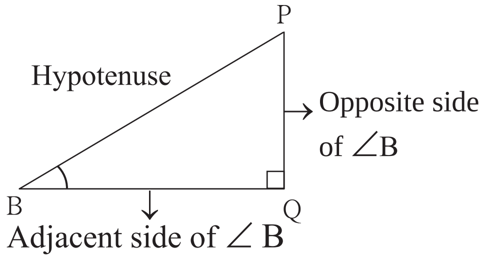


Fig.8.9

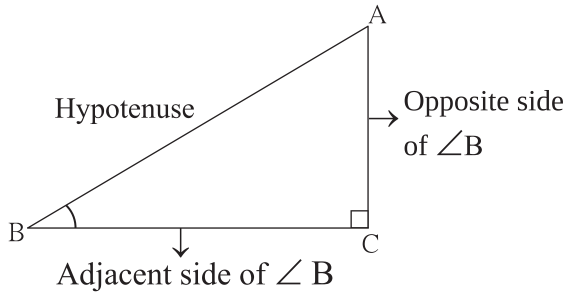


Fig.8.10

(i) In Δ PQB,

$$\frac{PQ}{PB} = \frac{\text{Opposite side of } \angle B}{\text{Hypotenuse}}$$

In Δ ACB,

$$\frac{AC}{AB} = \frac{\text{Opposite side of } \angle B}{\text{Hypotenuse}}$$

The ratios $\frac{PQ}{PB}$ and $\frac{AC}{AB}$ are equal.

$$\therefore \frac{PQ}{PB} = \frac{AC}{AB} = \frac{\text{Opposite side of } \angle B}{\text{Hypotenuse}}$$

This ratio is called the 'sine' ratio of $\angle$ B, and is written in brief as **'sin B'**.

(ii) In Δ PQB and Δ ACB,

$$\frac{BQ}{PB} = \frac{\text{Adjacent side of } \angle B}{\text{Hypotenuse}} \text{ and } \frac{BC}{AB} = \frac{\text{Adjacent side of } \angle B}{\text{Hypotenuse}}$$

$$\therefore \frac{BQ}{PB} = \frac{BC}{AB} = \frac{\text{Adjacent side of } \angle B}{\text{Hypotenuse}}$$

This ratio is called as the 'cosine' ratio of $\angle$ B , and written in brief as **'cos B'**

(iii) $$\frac{PQ}{BQ} = \frac{AC}{BC} = \frac{\text{Opposite side of } \angle B}{\text{Adjacent side of } \angle B}$$

This ratio is called as the tangent ratio of $\angle$ B, and written in brief as **tan B.**

Ex. :

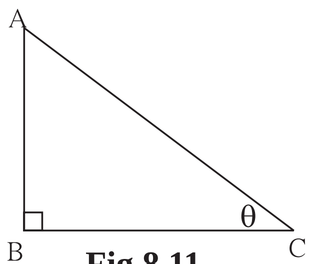


Fig.8.11

Sometimes we write measures of acute angles of a right angled triangle by using Greek letters θ (Theta), α (Alpha), β (Beta) etc.

In the adjacent figure of Δ ABC, measure of acute angle C is denoted by the letter θ. So we can write the ratios sin C, cos C, tan C as sin θ, cos θ, tan θ respectively.

$\sin C = \sin\theta = \dfrac{AB}{AC}$, $\cos C = \cos\theta = \dfrac{BC}{AC}$, $\tan C = \tan\theta = \dfrac{AB}{BC}$

Remember this !

- sin ratio $= \dfrac{\text{opposite side}}{\text{hypotenuse}}$
- cos ratio $= \dfrac{\text{adjacent side}}{\text{hypotenuse}}$
- tan ratio $= \dfrac{\text{opposite side}}{\text{adjcent side}}$
- $\sin\theta = \dfrac{\text{opposite side of } \angle\theta}{\text{hypotenuse}}$
- $\cos\theta = \dfrac{\text{adjacent side of } \angle\theta}{\text{hypotenuse}}$
- $\tan\theta = \dfrac{\text{opposite side of } \angle\theta}{\text{opposite side of } \angle\theta}$

Practice set 8.1

1. 

Fig. 8.12

In the Fig.8.12, $\angle R$ is the right angle of Δ PQR. Write the following ratios.
(i) sin P (ii) cos Q (iii) tan P (iv) tan Q

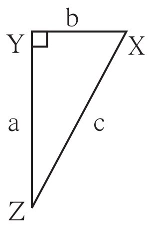
2. 

Fig. 8.13

In the right angled Δ XYZ, $\angle XYZ = 90^\circ$ and a,b,c are the lengths of the sides as shown in the figure. Write the following ratios,
(i) sin X (ii) tan Z (iii) cos X (iv) tan X.

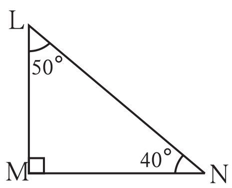
3. 

Fig. 8.14

In right angled Δ LMN, $\angle LMN = 90^\circ$
$\angle L = 50^\circ$ and $\angle N = 40^\circ$,
write the following ratios.
(i) sin 50° (ii) cos 50°
(iii) tan 40° (iv) cos 40°

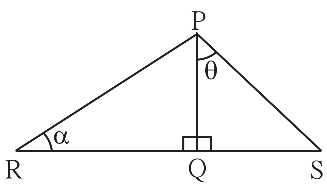
4. 

Fig. 8.15

In the figure 8.15,.$\angle PQR = 90^\circ$,
$\angle PQS = 90^\circ$, $\angle PRQ = \alpha$ and $\angle QPS = \theta$
Write the following trigonometric ratios.
(i) sin α, cos α, tan α
(ii) sin θ, cos θ, tan θ

Let's learn.

Relation among trigonometric ratios

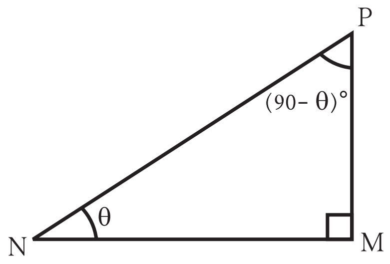


Fig.8.16

In the Fig.8.16

Δ PMN is a right angled triangle.

$\angle M = 90^\circ$, $\angle P$ and $\angle N$ are complimentary angles.

$\therefore$ If $\angle N = \theta$ then $\angle P = 90 - \theta$

$\sin\theta = \dfrac{PM}{PN}$ ....(1)

$\cos\theta = \dfrac{NM}{PN}$ ....(2)

$\tan\theta = \dfrac{PM}{NM}$ ....(3)

$\sin(90-\theta) = \dfrac{NM}{PN}$ ....(4)

$\cos(90-\theta) = \dfrac{PM}{PN}$ ....(5)

$\tan(90-\theta) = \dfrac{NM}{PM}$ ....(6)

$\therefore$ $\sin\theta = \cos(90-\theta)$ .... from (1) and (5)

$\cos\theta = \sin(90-\theta)$ .... from (2) and (4)

Also note that $\tan\theta \times \tan(90-\theta) = \dfrac{PM}{NM} \times \dfrac{NM}{PM}$ .... from (3) and (6)

$\therefore$ $\tan\theta \times \tan(90-\theta) = 1$

Similarly, $$\frac{\sin\theta}{\cos\theta} = \frac{\frac{PM}{PN}}{\frac{NM}{PN}} = \frac{PM}{PN} \times \frac{PN}{NM} = \frac{PM}{NM} = \tan\theta$$

Remember this !

$\cos(90-\theta) = \sin\theta$, $\sin(90-\theta) = \cos\theta$

$\dfrac{\sin\theta}{\cos\theta} = \tan\theta$, $\tan\theta \times \tan(90-\theta) = 1$

*** For more information**

$$\frac{1}{\sin\theta} = \operatorname{cosec}\theta, \quad \frac{1}{\cos\theta} = \sec\theta, \quad \frac{1}{\tan\theta} = \cot\theta$$

It means cosec θ, sec θ and cot θ are inverse ratios of sin θ, cos θ and tan θ respectively.

- $\sec\theta = \operatorname{cosec}(90 - \theta)$
- $\operatorname{cosec}\theta = \sec(90 - \theta)$
- $\tan\theta = \cot(90 - \theta)$
- $\cot\theta = \tan(90 - \theta)$

Let's recall.

Theorem of 30°- 60°-90° triangle :

We know that if the measures of angles of a triangle are 30°,60°, 90° then side opposite to 30° angle is half of the hypotenuse and side opposite to 60° angle is $\frac{\sqrt{3}}{2}$ of hypotenuse.

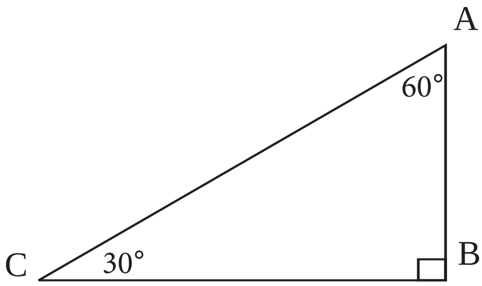


Fig. 8.17

In the Fig. 8.17, Δ ABC is a right angled triangle. ∠C = 30°, ∠A = 60°, ∠B = 90° .

$$\therefore \ AB = \frac{1}{2}AC \text{ and } BC = \frac{\sqrt{3}}{2}AC$$

Let's learn.

Trignometric ratios of 30° and 60° angles

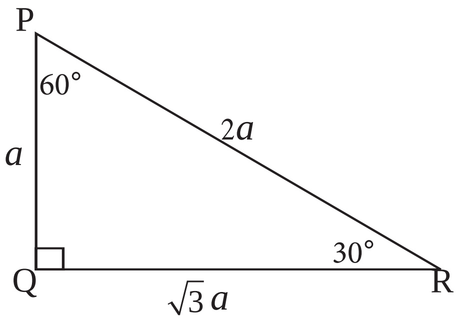


Fig. 8.18

In right angled Δ PQR if ∠R = 30°, ∠P = 60°, ∠Q = 90° and PQ = a

then $PQ = \frac{1}{2} PR$

$a = \frac{1}{2} PR$

$\therefore \quad PR = 2a$

$QR = \frac{\sqrt{3}}{2} PR$

$QR = \frac{\sqrt{3}}{2} \times 2a$

$QR = \sqrt{3}\,a$

$\therefore$ If $PQ = a$, then $PR = 2a$ and $QR = \sqrt{3}\,a$

(I) Trigonometric ratios of the 30° angle

$$\sin 30^\circ = \frac{PQ}{PR} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{QR}{PR} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{PQ}{QR} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

(II) Trigonometric ratios of 60° angle

$$\sin 60^\circ = \frac{QR}{PR} = \frac{\sqrt{3}\,a}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{PQ}{PR} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{QR}{PQ} = \frac{\sqrt{3}\,a}{a} = \sqrt{3}$$

In right angled Δ PQR, $\angle Q = 90^\circ$. Therefore $\angle P$ and $\angle R$ are complimentary angles of each other. Verify the relation between sine and cosine ratios of complimentary angles here also.

$\sin \theta = \cos (90 - \theta)$

$\sin 30^\circ = \cos (90^\circ - 30^\circ) = \cos 60^\circ$

$\sin 30^\circ = \cos 60^\circ$

$\cos \theta = \sin (90 - \theta)$

$\cos 30^\circ = \sin (90^\circ - 30^\circ) = \sin 60^\circ$

$\cos 30^\circ = \sin 60^\circ$

Remember this !

$\sin 30^\circ = \frac{1}{2}$	$\cos 30^\circ = \frac{\sqrt{3}}{2}$	$\tan 30^\circ = \frac{1}{\sqrt{3}}$
$\sin 60^\circ = \frac{\sqrt{3}}{2}$	$\cos 60^\circ = \frac{1}{2}$	$\tan 60^\circ = \sqrt{3}$

(III) Trigonometric ratios of the 45° angle

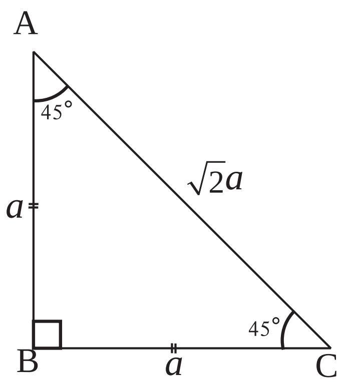


Fig.8.19

In right angled Δ ABC, $\angle B = 90^\circ$, $\angle A = 45^\circ$, $\angle C = 45^\circ$ $\therefore$ This is an isosceles triangle.

Suppose AB = a then BC = a.

Using Pythagoras' theorem, let us find the length of AC.

$AC^2 = AB^2 + BC^2$

$= a^2 + a^2$

$AC^2 = 2a^2$

$\therefore$ AC = $\sqrt{2}\,a$

In the Fig. 8.19, $\angle C = 45^\circ$

$$\sin 45^\circ = \frac{AB}{AC} = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{BC}{AC} = \frac{a}{\sqrt{2}\,a} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{AB}{BC} = \frac{a}{a} = 1$$

Remember this !

$\sin 45^\circ = \frac{1}{\sqrt{2}}$, $\cos 45^\circ = \frac{1}{\sqrt{2}}$, $\tan 45^\circ = 1$

(IV) Trigonometric ratios of the angle 0° and 90°

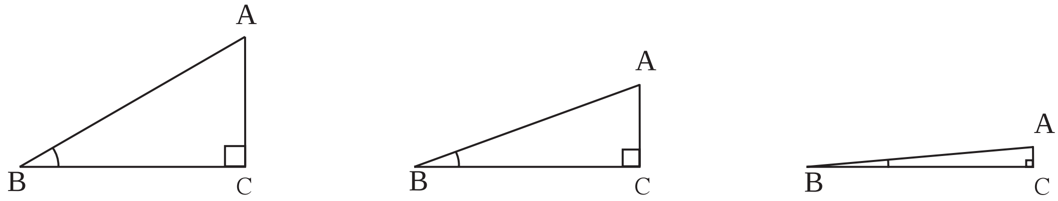


Fig.8.20

In the right angled Δ ACB, $\angle C = 90^\circ$ and $\angle B = 30^\circ$. We know that $\sin 30^\circ = \frac{AC}{AB}$. Keeping the length of side AB constant, if the measure of $\angle B$ goes on decreasing the length of AC, which is opposite to $\angle B$ also goes on decreasing. So as the measure of $\angle B$ decreases, then value of $\sin\theta$ also decreases.

$\therefore$ when measure of $\angle B$ becomes 0°, then length of AC becomes 0.

$\therefore \sin 0^\circ = \frac{AC}{AB} = \frac{0}{AB} = 0 \qquad \therefore \sin 0^\circ = 0$

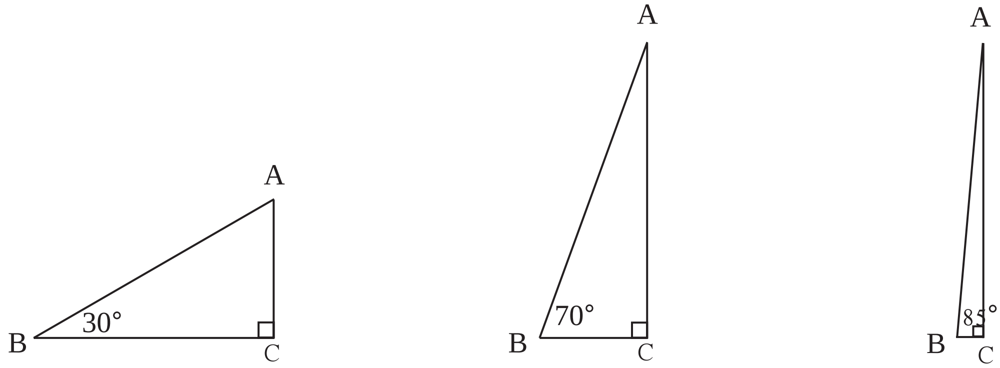


Fig.8.21

Now look at the Fig. 8.21. In this right angled triangle, as the measure of $\angle$B increases the length of AC also increases. When measure of $\angle$B becomes 90°, the length of AC become equal to AB.

$$\therefore \sin 90^\circ = \frac{AC}{AB} \qquad \therefore \sin 90^\circ = 1$$

We know the relations between trigonometric ratios of complimentary angles.

$$\sin \theta = \cos (90 - \theta) \quad \text{and} \quad \cos \theta = \sin (90 - \theta)$$

$$\therefore \cos 0^\circ = \sin (90 - 0)^\circ = \sin 90^\circ = 1$$

$$\text{and } \cos 90^\circ = \sin (90 - 90)^\circ = \sin 0^\circ = 0$$

Remember this !

$\sin 0^\circ = 0$, $\sin 90^\circ = 1$, $\cos 0^\circ = 1$, $\cos 90^\circ = 0$

We know that,

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \therefore \tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

$$\text{But } \tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}$$

But we can not do the division of 1 by 0. Note that θ is an acute angle. As it increases and reaches the value of 90°, $\tan \theta$ also increases indefinitely. Hence we can not find the definite value of tan 90.

Remember this !

Trigonometric ratios of particular ratios.

Ratios \ Measures of angles	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

Solved Examples :

Ex. (1) Find the value of 2tan 45° + cos 30° – sin 60°

Solution : 2tan 45° + cos 30° – sin 60°

$$= 2 \times 1 + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$

$= 2 + 0$

$= 2$

Ex. (2) Find the value of $\frac{\cos 56°}{\sin 34°}$

Solution : 56° + 34° = 90° means 56 and 34 are the measures of complimentary angles.

sin θ = cos (90- θ)

$\therefore$ sin 34° = cos (90- 34)° = cos 56°

$$\therefore \quad \frac{\cos 56°}{\sin 34°} = \frac{\cos 56°}{\cos 56°} = 1$$

Ex. 3 In right angled Δ ACB, If ∠C = 90°, AC = 3, BC = 4.

Find the ratios sin A, sin B, cos A, tan B

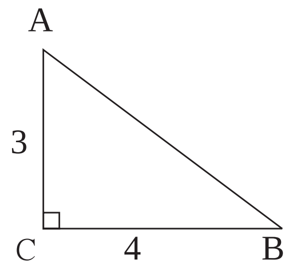


Fig. 8.22

Solution : In right angled Δ ACB, using Pythagoras' theorem,

$AB^2 = AC^2 + BC^2$

$= 3^2 + 4^2 = 5^2$

$\therefore$ AB = 5

$$\sin A = \frac{BC}{AB} = \frac{4}{5} \qquad \cos A = \frac{AC}{AB} = \frac{3}{5}$$

$$\text{and } \sin B = \frac{AC}{AB} = \frac{3}{5} \qquad \tan B = \frac{AC}{BC} = \frac{3}{4}$$

Ex. 4 In right angled triangle Δ PQR, ∠Q = 90°, ∠R= θ and if sin θ = $\frac{5}{13}$ then find cos θ and tan θ.

Solution :

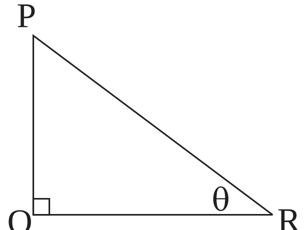


Fig. 8.23

In right angled Δ PQR, ∠R= θ

$\sin \theta = \frac{5}{13}$

$$\therefore \quad \frac{PQ}{PR} = \frac{5}{13}$$

$\therefore$ Let PQ = $5k$ and PR = $13k$

Let us find QR by using Pythagoras' theorem,

$$PQ^2 + QR^2 = PR^2$$

$$(5k)^2 + QR^2 = (13k)^2$$

$$25k^2 + QR^2 = 169\,k^2$$

$$QR^2 = 169\,k^2 - 25k^2$$

$$QR^2 = 144\,k^2$$

$$QR = 12k$$

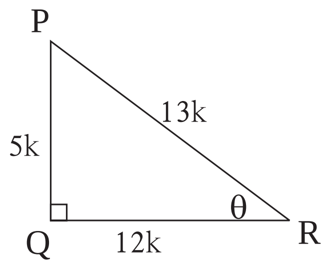


Fig. 8.24

Now, in right angled Δ PQR, PQ = $5k$, PR = $13k$ and QR = $12k$

$$\therefore \cos\theta = \frac{QR}{PR} = \frac{12k}{13k} = \frac{12}{13}, \tan\theta = \frac{PQ}{QR} = \frac{5k}{12k} = \frac{5}{12}$$

Use your brain power!

(1) While solving above example, why the lengths of PQ and PR are taken $5k$ and $13k$?

(2) Can we take the lengths of PQ and PR as 5 and 13 ? If so then what changes are needed in the writing of the solution.

Important Equation in Trigonometry

Δ PQR is a right angled triangle.

$\angle PQR = 90^\circ$, $\angle R = \theta$

$$\sin\theta = \frac{PQ}{PR} \quad \text{..........(I)}$$

and $$\cos\theta = \frac{QR}{PR} \quad \text{..........(II)}$$

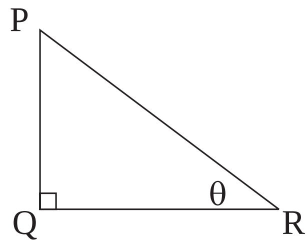


Fig.8.25

Using Pythagoras' theorem,

$PQ^2 + QR^2 = PR^2$

$$\therefore \frac{PQ^2}{PR^2} + \frac{QR^2}{PR^2} = \frac{PR^2}{PR^2}$$ .... dividing each term by PR^2

$$\therefore \left(\frac{PQ}{PR}\right)^2 + \left(\frac{QR}{PR}\right)^2 = 1$$

$\therefore$ $(\sin\theta)^2 + (\cos\theta)^2 = 1$ ..from (I) & (II)

Remember this !

'Square of' sin θ means $(\sin\theta)^2$. It is written as $\sin^2\theta$.

We have proved the equation $\sin^2\theta + \cos^2\theta = 1$ using Pythagoras' theorem, where θ is an acute angle of a right angled triangle.

Verify that the equation is true even when $\theta = 0^\circ$ or $\theta = 90^\circ$.

Since the equation $\sin^2\theta + \cos^2\theta = 1$ is true for any value of θ. So it is a basic trigonometrical identity.

(i) $0 \le \sin\theta \le 1, \quad 0 \le \sin^2\theta \le 1$ (ii) $0 \le \cos\theta \le 1, \quad 0 \le \cos^2\theta \le 1$

Practice set 8.2

1. In the following table, a ratio is given in each column. Find the remaining two ratios in the column and complete the table.

sin θ		$\frac{11}{61}$		$\frac{1}{2}$				$\frac{3}{5}$	
cos θ	$\frac{35}{37}$				$\frac{1}{\sqrt{3}}$				
tan θ			1			$\frac{21}{20}$	$\frac{8}{15}$		$\frac{1}{2\sqrt{2}}$

2. Find the values of -

(i) $5\sin 30^\circ + 3\tan 45^\circ$

(ii) $\frac{4}{5}\tan^2 60^\circ + 3\sin^2 60^\circ$

(iii) $2\sin 30^\circ + \cos 0^\circ + 3\sin 90^\circ$

(iv) $\frac{\tan 60}{\sin 60 + \cos 60}$

(v) $\cos^2 45^\circ + \sin^2 30^\circ$

(vi) $\cos 60^\circ \times \cos 30^\circ + \sin 60^\circ \times \sin 30^\circ$

3. If $\sin\theta = \frac{4}{5}$ then find $\cos\theta$

4. If $\cos\theta = \frac{15}{17}$ then find $\sin\theta$

Problem set 8

1. Choose the correct alternative answer for following multiple choice questions.

(i) Which of the following statements is true ?

(A) $\sin \theta = \cos (90 - \theta)$ (B) $\cos \theta = \tan (90 - \theta)$

(C) $\sin \theta = \tan (90 - \theta)$ (D) $\tan \theta = \tan (90 - \theta)$

(ii) Which of the following is the value of $\sin 90^\circ$?

(A) $\frac{\sqrt{3}}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) 1

(iii) $2 \tan 45^\circ + \cos 45^\circ - \sin 45^\circ =$?

(A) 0 (B) 1 (C) 2 (D) 3

(iv) $\frac{\cos 28^\circ}{\sin 62^\circ} =$?

(A) 2 (B) -1 (C) 0 (D) 1

2. In right angled Δ TSU, TS = 5, $\angle S = 90^\circ$, SU = 12 then find sin T, cos T, tan T. Similarly find sin U, cos U, tan U.

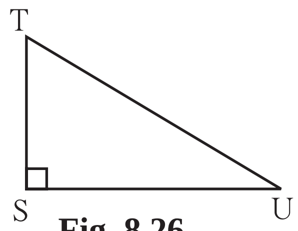


Fig. 8.26

3. In right angled Δ YXZ, $\angle X = 90^\circ$, XZ = 8 cm, YZ = 17 cm, find sin Y, cos Y, tan Y, sin Z, cos Z, tan Z.

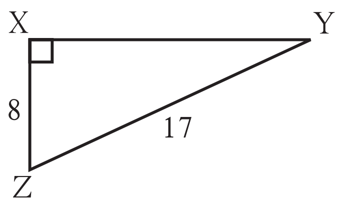


Fig. 8.27

4. In right angled Δ LMN, if $\angle N = \theta$, $\angle M = 90^\circ$, $\cos \theta = \frac{24}{25}$, find $\sin \theta$ and $\tan \theta$

Similarly, find $(\sin^2 \theta)$ and $(\cos^2 \theta)$.

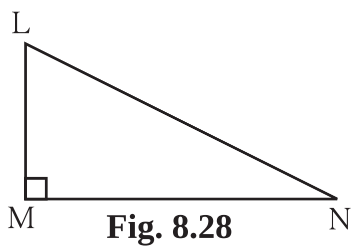


Fig. 8.28

5. Fill in the blanks.

(i) $\sin 20^\circ = \cos \square^\circ$

(ii) $\tan 30^\circ \times \tan \square^\circ = 1$

(iii) $\cos 40^\circ = \sin \square^\circ$